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# Weak compactness in the classes of strengthened $\sigma$-order continuous linear functionals of Riesz spaces

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ABSTRACT

Let $L$ be an Archimedean Riesz space and let $\mathscr{A}(L)$ be the set of all strengthened $\sigma$-order continuous linear functionals defined on $L$. Veksler gave some characterizations of these functionals in [11]. The purpose of this paper is to characterize the weakly compact subsets of $\mathscr{A}(L)$ in terms of the order structure of $L$ and also generalize the result (Theorem 2.5) given by Dashiell in [4].

## 1. INTRODUCTION

Throughout this paper we assume that every Riesz space considered is Archimedean. For notation and basic terminology concerning Riesz spaces not explained below, see [1] and [12].

(1) The following result which will be useful later was given by Quinn [9].

THEOREM A. Let $L$ be a Riesz space. Then there exists an essentially unique smallest $\sigma$-Dedekind complete Riesz space $L^\sigma$ in which $L$ can be embedded as an order dense Riesz subspace and which has the property that the ideal generated by $L$ in $L^\sigma$ is $L^\sigma$ itself. (See [9] for related results.)

$L^\sigma$ is called *$\sigma$-Dedekind completion* of $L$.

(2) Let $L$ be a Riesz space. $L^\sim$ denotes the order dual of $L$. $L_c^\sim$ and $L_n^\sim$ denote the band of $\sigma$-order continuous and order continuous linear functionals in $L^\sim$ respectively. A positive functional $\phi$ in $L^\sim$ is called *strengthened $\sigma$-order continuous* if for any sequence $\theta \le x_n \downarrow$ in $L$,

$$\inf\{\phi(x_n)\} = \sup\{\phi(y) \mid y \in \mathscr{L}(\{x_n\})\},$$

where $\mathscr{L}(\{x_n\})$ denotes $\{y \in L \mid \theta \le y \le x_n$ for all $n\}$. A $\phi \in L^\sim$ is called *strengthened $\sigma$-order continuous* if $\phi^+$ and $\phi^-$ are both strengthened $\sigma$-order continuous. We denote by $\mathscr{A}(L)$ the set of all strengthened $\sigma$-order continuous functionals in $L^\sim$. Note that $\mathscr{A}(L)$ is a band of $L^\sim$ and $L_n^\sim \subset \mathscr{A}(L) \subset L_c^\sim$ (Assertion 1.2 of [11]). If $L$ is $\sigma$-Dedekind complete, then it follows immediately that $\mathscr{A}(L) = L_c^\sim$.

Let $L^\sigma$ be the $\sigma$-Dedekind completion of $L$. Put $G = \{u \in L^\sigma \mid u = \sup\{x_n\}, x_n \in L\}$ and $H = \{v \in L^\sigma \mid -v \in G\}$. Let $\theta \le \phi \in \mathscr{A}(L)$. For each $u \in G$ define $\bar{\phi}$ by $\bar{\phi}(u) = \sup\{\phi(x_n)\}$, where $u = \sup\{x_n\}, x_n \in L$, and for each $v \in H$ define $\bar{\phi}$ (denoted by $\bar{\phi}$ again) by $\bar{\phi}(v) = -\bar{\phi}(-v)$. Let $M$ denote the set of all elements $w$ of $L^\sigma$ such that for any $\varepsilon > 0$ there exist $u \in G$ and $v \in H$ satisfying $v \le w \le u$ and $\bar{\phi}(u) - \bar{\phi}(v) < \varepsilon$. For each $w \in M$ it is clear that $\sup\{\phi(v) \mid v \le w, v \in H\} = \inf\{\phi(u) \mid w \le u, u \in G\}$. We now define $\bar{\phi}$ on $M$ by $\bar{\phi}(w) = \sup\{\phi(v) \mid v \le w, v \in H\}$ for $w \in M$. It then follows that $\bar{\phi}$ is a unique positive linear extension of $\phi$ on $M$ and $M$ is a $\sigma$-Dedekind complete Riesz subspace of $L^\sigma$. (Use the arguments given in 12A-12G of [8].) Furthermore, by Theorem A $M = L^\sigma$ holds. Thus we have the following result given by Veksler in [11].

THEOREM B. Let $L$ be a Riesz space and $L^\sigma$ its $\sigma$-Dedekind completion. Then every $\theta \le \phi \in \mathscr{A}(L)$ has a unique positive linear extension $\bar{\phi}$ on $L^\sigma$ and $\bar{\phi} \in (L^\sigma)_c^\sim$. (See [11] for related results.)

For each $\phi \in \mathscr{A}(L)$ put $\phi = \phi^+ - \phi^-$ and define $\bar{\phi}$ on $L^\sigma$ by $\bar{\phi} = \overline{\phi^+} - \overline{\phi^-}$. For every $\theta \le \phi_1, \phi_2 \in \mathscr{A}(L)$ it is simple to verify that $\overline{\phi_1 + \phi_2} = \overline{\phi_1} + \overline{\phi_2}$ on $L^\sigma$. Hence the extension mapping: $\phi \to \bar{\phi}$ is well-defined and positive. The next corollary is the supplement of Theorem 2.1 given in [11].

COROLLARY 1. (1) Each $\phi \in \mathscr{A}(L)$ has a unique $\sigma$-order continuous extension $\bar{\phi}$ on $L^\sigma$ and the extension mapping: $\phi \to \bar{\phi}$ is a Riesz isomorphism of $\mathscr{A}(L)$ into $(L^\sigma)_c^\sim$.

(2) If $\phi \in (L^\sigma)^\sim$ and $\phi \mid L \in \mathscr{A}(L)$, where $\phi \mid L$ denotes the restriction of $\phi$ on $L$, then $\phi \in (L^\sigma)_c^\sim$ holds.

The proof is left to the reader.

## 2. COMPACTNESS IN $\mathscr{A}(L)$

In [1] Aliprantis and Burkinshaw summarized the characterization of weakly compact subsets of $L^\sim$, $L_c^\sim$ and $L_n^\sim$ in terms of the order structure on $L$. Similarly, in this section we will give some information about weak compactness in $\mathscr{A}(L)$.

Let $L$ be a Riesz space and $A$ a $\sigma(L^\sim, L)$-bounded subset of $L^\sim$. For each $x \in L$ define $\varrho_A$ by $\varrho_A(x) = \sup\{|\phi(x)| \mid \phi \in A\}$. $\varrho_A$ is a seminorm of $L$, and if $A$ is a solid of $L^\sim$, then $\varrho_A$ is a Riesz seminorm. A sequence $\{x_n\}$ of $L$ is called

$\varrho_A$-*Cauchy* if for any $\varepsilon>0$ there exists $n_0$ such that $\varrho_A(x_n-x_m)<\varepsilon$ for all $n,m\geq n_0$. A subset $A$ of $L^\sim$ is called *order-equicontinuous* on $L$ if $\{x_n\}\subset L$ is a $\varrho_A$-Cauchy sequence whenever $\theta\leq x_n\uparrow\leq x$ holds in $L$.

For a subset $A$ of $\mathscr{A}(L)$ we denote by $\bar{A}$ the set $\{\bar{\phi}\in(L^\sigma)_c^\sim \mid \phi\in A\}$. By Corollary 1 it is clear that $A$ is $|\sigma|(\mathscr{A}(L),L)$-bounded if and only if $\bar{A}$ is $|\sigma|((L^\sigma)_c^\sim,L^\sigma)$-bounded.

THEOREM 2. Let $L$ be a Riesz space and $L^\sigma$ its $\sigma$-Dedekind completion. Then for a subset $A$ of $\mathscr{A}(L)$ the following statements are equivalent.

(1) The solid hull $S$ of $A$ is $\sigma(\mathscr{A}(L),L)$-relatively compact.
(2) $\lim \varrho_{\bar{A}}(u_n)=0$ whenever $u_n\downarrow\theta$ in $L^\sigma$.
(3) $\bar{A}$ is order-equicontinuous on $L^\sigma$.
(4) $A$ is order-equicontinuous on $L$.

PROOF. (1)$\to$(2) Note first that $S$ is $|\sigma|(\mathscr{A}(L),L)$-bounded and hence $\bar{S}$ is also $|\sigma|((L^\sigma)_c^\sim,L^\sigma)$-bounded. Now, assume that (2) does not hold. Then there exist $\varepsilon>0$ and $u_n\downarrow\theta$ in $L^\sigma$ such that $\varrho_{\bar{A}}(u_n)>2\varepsilon$ for all $n$. For each $n$ choose $\phi_n\in A$ with $|\overline{\phi_n}(u_n)|>2\varepsilon$ and let $\theta\leq\Phi\in(L^\sigma)^\sim$ be a $\sigma((L^\sigma)^\sim,L^\sigma)$-accumulation point of $\{|\overline{\phi_n}|\}$. Then $\Phi\mid L$ is a $\sigma(L^\sim,L)$-accumulation point of $\{|\phi_n|\}$. Since $\{|\phi_n|\}\subset S$, it is clear that $\Phi\mid L\in\mathscr{A}(L)$. Hence it follows by Corollary 1 that $\Phi\in(L^\sigma)_c^\sim$. For each $n$ choose $m\geq n$ such that $|(|\overline{\phi_m}|-\Phi)(u_n)|<\varepsilon$. Then it follows that

$$\Phi(u_n)=|\overline{\phi_m}|(u_n)-|(|\overline{\phi_m}|-\Phi)(u_n)|\geq|\overline{\phi_m}|(u_n)-\varepsilon\geq|\overline{\phi_m}|(u_m)-\varepsilon>\varepsilon.$$

But this is a contradiction, and hence (1) implies (2).

(2)$\to$(3) see Theorem 20.7 of [1] and (3)$\to$(4) is obvious.

(4)$\to$(1) Since $\mathscr{A}(L)$ is the topological dual of $(L,|\sigma|(L,\mathscr{A}(L)))$, by Theorem 20.5 of [1] $S$ is order-equicontinuous in $L$ and hence $|\sigma|(\mathscr{A}(L),L)$-bounded. Furthermore, it follows by Theorem 20.1 of [1] that $S$ is $\sigma(L^\sim,L)$-relatively compact. Now, we have to show that the $\sigma(L^\sim,L)$-closure of $S$ lies in $\mathscr{A}(L)$. To this end let $\psi$ be in $\sigma(L^\sim,L)$-closure of $S$. For $\theta\leq x_n\downarrow$ in $L$ replace $\mathscr{L}(\{x_n\})$ by a net $\{y_\alpha\}$ with $\theta\leq y_\alpha\uparrow\leq x_n$ for all $n$ and put $z_{n,\alpha}=x_n-y_\alpha$ for each $(n,\alpha)$. Since $S$ is order-equicontinuous, $z_{n,\alpha}\downarrow\theta$ in $L$ implies that $\{z_{n,\alpha}\}$ is $\varrho_S$-Cauchy net by Theorem 10.1 of [1]. Hence for any $\varepsilon>0$ there exists $(n_0,\alpha_0)$ such that $\varrho_S(z_{n,\alpha}-z_{m,\beta})<\varepsilon$ for $(n,\alpha)$, $(m,\beta)\geq(n_0,\alpha_0)$. For fixed $(n,\alpha)\geq(n_0,\alpha_0)$ choose $\phi\in S$ such that $|(\phi-\psi)(z_{n,\alpha})|<\varepsilon$. Then for any $(m,\beta)\geq(n,\alpha)$ it follows that

$$\begin{aligned}|\psi(z_{n,\alpha})|&\leq|(\phi-\psi)(z_{n,\alpha})|+|\phi(z_{n,\alpha})|<\varepsilon+|\phi(z_{n,\alpha})-\phi(z_{m,\beta})|\\&\quad+|\phi(z_{m,\beta})|<2\varepsilon+|\phi(z_{m,\beta})|.\end{aligned}$$

Since $\phi(z_{n,\alpha})\xrightarrow[(n,\alpha)]{}0$, we have $|\psi(z_{n,\alpha})|\leq2\varepsilon$ for all $(n,\alpha)\geq(n_0,\alpha_0)$. Hence it follows that $\psi\in\mathscr{A}(L)$. Thus $S$ is $\sigma(\mathscr{A}(L),L)$-relatively compact and the proof is complete.

The following result shows that $\sigma(\mathscr{A}(L), L)$-relative compactness and the concepts of order-equicontinuity in $\mathscr{A}(L)$ coincide for uniformly complete Riesz spaces.

THEOREM 3. Let $L$ be a uniformly complete Riesz space and $L^\sigma$ its $\sigma$-Dedekind completion. Then for a subset $A$ of $\mathscr{A}(L)$ the following statements are equivalent:

(1) $A$ is $\sigma(\mathscr{A}(L), L)$-relatively compact.
(2) $\bar{A}$ is $\sigma((L^\sigma)_c^\sim, L^\sigma)$-relatively compact.
(3) $\bar{A}$ is order-equicontinuous on $L^\sigma$.
(4) $A$ is order-equicontinuous on $L$.

PROOF. (1)$\rightarrow$(2) Since $\mathscr{A}(L)$ is the topological dual of $(L, |\sigma|(L, \mathscr{A}(L)))$, by Theorem 19.15 of [1] $A$ is $|\sigma|(\mathscr{A}(L), L)$-bounded and hence $\bar{A}$ is also $|\sigma|((L^\sigma)_c^\sim, L^\sigma)$-bounded. Thus it follows by Theorem 20.1 of [1] that $\bar{A}$ is $\sigma((L^\sigma)^\sim, L^\sigma)$-relatively compact. To complete the proof we shall show that the $\sigma((L^\sigma)^\sim, L^\sigma)$-closure of $\bar{A}$ lies in $(L^\sigma)_c^\sim$. To this end let $\Phi$ be in $\sigma((L^\sigma)^\sim, L^\sigma)$-closure of $\bar{A}$. If $\Phi$ is a $\sigma((L^\sigma)^\sim, L^\sigma)$-accumulation point of $\bar{A}$, then it is simple to verify that $\Phi \mid L$ is also a $\sigma(\mathscr{A}(L), L)$-accumulation point of $A$. Hence $\Phi \mid L \in \mathscr{A}(L)$ holds and it follows by Corollary 1 that $\Phi \in (L^\sigma)_c^\sim$. This means that the $\sigma((L^\sigma)^\sim, L^\sigma)$-closure of $\bar{A}$ lies in $(L^\sigma)_c^\sim$.

(2)$\rightarrow$(3) see Theorem 20.9 of [1].

(3)$\rightarrow$(4) is obvious and (4)$\rightarrow$(1) follows from Theorem 2.

The mapping $x \rightarrow \tilde{x}$ of $L$ into $(L^\sim)_n^\sim$ defined by $\tilde{x}(\phi) = \phi(x)$ for $\phi \in L^\sim$ is a Riesz homomorphism. Let $I$ denote the ideal generated by the image of $L$ into $(L^\sim)_n^\sim$. By once more applying the definition above to the pair $L^\sim$ and $I$ we have that the mapping of $L^\sim$ into $I_n^\sim$ is a Riesz isomorphism, and by the argument given in Theorem 20.13 of [1] it follows that the mapping $\phi \rightarrow \tilde{\phi}$ is a Riesz isomorphism of $L^\sim$ onto $I_n^\sim$, where $\tilde{\phi}(f) = f(\phi)$ for $f \in I$. We now have the following result.

THEOREM 4. Let $L$ be a uniformly complete Riesz space and let $I$ be the ideal generated by the image of $L$ in $(L^\sim)_n^\sim$. Then for a subset $A$ of $\mathscr{A}(L)$ the following statements are equivalent:

(1) $A$ is order-equicontinuous on $L$.
(2) $A$ is $\sigma(\mathscr{A}(L), I)$-relatively compact.
(3) $A$ is $\sigma(\mathscr{A}(L), L)$-relatively compact.

PROOF. (1)$\rightarrow$(2) Let $\theta \leq f \in I$ and $\varepsilon > 0$. Choose an $\theta \leq x \in L$ with $f \leq \tilde{x}$. Since $\mathscr{A}(L)$ is the topological dual of $(L, |\sigma|(L, \mathscr{A}(L)))$, by Theorem 20.6 of [1] there exists $\theta \leq \psi \in \mathscr{A}(L)$ such that $(|\phi| - \psi)^+(x) < \varepsilon$ for all $\phi \in A$. Hence $(|\phi| - \psi)^+(f) \leq \varepsilon$ holds for all $\phi \in A$, and by Theorem 20.6 of [1] again $A$ is order-equicontinuous on $I$. Consequently it follows from Theorem 20.11 of [1] that $A$ is $\sigma(I_n^\sim, I)$-relatively compact. Since $I_n^\sim = \mathscr{A}(L) \oplus \mathscr{A}(L)^{\mathrm{d}} (= L^\sim)$, it is simple to verify that the $\sigma(I_n^\sim, I)$-closure of $A$ in $I_n^\sim$ lies in $\mathscr{A}(L)$. Hence $A$ is $\sigma(\mathscr{A}(L), I)$-relatively compact.

$(2) \to (3)$ is obvious and $(3) \to (1)$ follows from Theorem 3.

### 3. ORDER-CAUCHY COMPLETE RIESZ SPACES AND $l^\infty$ SPACES

Let $L$ be an order-Cauchy complete Riesz space satisfying $L_c^\sim = \mathscr{A}(L)$. In this section we will relate $\sigma(L^\sim, L)$-convergent sequence of $L^\sim$ with the concept of order-equicontinuity and also generalize the main result given by Dashiell in [4].

A sequence $\{x_n\}$ in Riesz space $L$ is called *order-Cauchy* if there exists a sequence $\{e_n\}$ of $L^+$ with $e_n \downarrow \theta$ such that $|x_m - x_n| < e_n$ for all $m > n$. A sequence $\{x_n\}$ in $L$ *order-converges* to $x$ in $L$ if there exists a sequence $\{e_n\}$ of $L^+$ with $e_n \downarrow \theta$ and $|x_n - x| \leq e_n$ for all $n$. A Riesz space $L$ is called *order-Cauchy complete* if every order-Cauchy sequence in $L$ order-converges in $L$ to some element of $L$.

DEFINITION 5. A Riesz space $L$ is called to have the *strengthened $\sigma$-order (s.$\sigma$-order) continuity property* if $L_c^\sim = \mathscr{A}(L)$.

We start with the following lemma which will be useful later.

LEMMA 6. Let $L$ be a Riesz space and $\theta < \phi \in \mathscr{A}(L)$. Then for every $x \in L$ with $x \geq \theta$ and $\phi(x) = 0$, there exists $\theta \leq y \in L$ such that $x \wedge y = \theta$ and $\phi(y) > 0$.

PROOF. Let $\theta \leq \phi \in \mathscr{A}(L)$. Assume that $\phi(x) = 0$ for an $x \geq \theta$ in $L$. Choose $\theta \leq z \in L$ with $\phi(z) > 0$ and put $z_n = z - z \wedge nx$ for each $n$. Since $\theta \leq z_n \downarrow$ in $L$ and $\phi(z_n) = \phi(z)$ for all $n$, for any $\varepsilon > 0$ there exists $y \in \mathscr{L}(\{z_n\})$ such that $\phi(z) - \varepsilon < \phi(y)$. Choose a $y \in \mathscr{L}(\{z_n\})$ with $\phi(y) > 0$. Clearly, $x \wedge y \leq z_n \leq z - x \wedge y$ holds for all $n$. Hence it follows that $2(x \wedge y) \leq z$. Similarly, we have $n(x \wedge y) \leq z$ for all $n$. Thus $x \wedge y = \theta$ holds in $L$ and $\phi(y) > 0$.

We now give one of the main results of this section.

THEOREM 7. Let $L$ be an order-Cauchy complete Riesz space with the s.$\sigma$-order continuity property. Then every $\sigma(L^\sim, L)$-Cauchy sequence in $L^\sim$ is order-equicontinuous on $L$.

PROOF. *Part* 1. Assume that $\{\phi_n\}$ is $\sigma(L^\sim, L)$-Cauchy sequence, but not order-equicontinuous on $L$. Since $L$ is uniformly complete, by Theorem 19.15 of [1] $\{\phi_n\}$ is $|\sigma|(L^\sim, L)$-bounded. Using Theorem 20.6 of [1] there exist $\varepsilon > 0$, an order bounded disjoint sequence $\{e_n\} \subset [\theta, e]$ and a subsequence $\{\phi_{k_n}\}$ of $\{\phi_n\}$ such that $|\phi_{k_n}(e_n)| > 2\varepsilon$ for each $n$. For simplicity of notation, let the original sequences have this property: $|\phi_n(e_n)| > 2\varepsilon$. Denote by $I_n$ the principal ideal generated by $e_n$. Then for each $n$ there exists $\theta \leq \psi_n \in L^\sim$ such that $\psi_n(x) = |\phi_n|(x)$ for $x \in I_n$ and $\psi_n(x) = 0$ for $x \in I_n^{\mathrm{d}}$, where $\psi_n(x) = \sup |\phi_n|(x \wedge m e_n)$ for $\theta \leq x \in L$. (See Theorem 83.8 of [12]) Clearly, $\{\psi_n\}$ is $|\sigma|(L^\sim, L)$-bounded.

Since $\theta \leq x_n = \sum_{k=1}^{n} e_k \uparrow \leq e$ in $L$, it is immediate that $\psi_n(e) \geq \psi_n(x_n) = \psi_n(e_n) > 2\varepsilon$.

Let $\phi_0$ be a $\sigma(L^{\sim}, L)$-accumulation point of $\{\psi_n\}$. For each $n$ and any $\delta > 0$ there exists some $m > n$ such that $|(\phi_0 - \psi_m)(x_n)| < \delta$. Hence $\phi_0(x_n) = 0$ holds for all $n$, but $\phi_0(e) > \varepsilon$. We now assume that $\phi_0 \in L_c^{\sim}$, and put $u = \sup\{x_n\}$ in $L^\sigma$. Since $L_c^{\sim} = \mathscr{A}(L)$, it is immediate that $\bar{\phi}_0(u) = \lim \phi_0(x_n) = 0$. By Lemma 6 choose a $v \in L^\sigma$ with $u \wedge v = \theta$ and $\bar{\phi}_0(v) > 0$. Then there exists some $x \in L$ such that $\theta \leq x \leq v$ in $L^\sigma$ and $\bar{\phi}_0(x) = \phi_0(x) > 0$. Since $x_n \wedge x = \theta$ in $L$ holds for each $n$, it is clear that $\psi_n(x) = 0$ for all $n$. From this we have $\phi_0(x) = 0$, which contradicts $\phi_0(x) > 0$. Thus $\phi_0 \neq 0$ is not $\sigma$-order continuous.

*Part* 2. Consequently, there exists a sequence $\{y_n\}$ in $L^+$ such that $y_n \downarrow \theta$ in $L$, but for some $\varepsilon' > 0$, $\phi_0(y_n) > 4\varepsilon'$ for all $n$. Since $\phi_0$ is $\sigma(L^{\sim}, L)$-accumulation point of $\{\psi_n\}$, we can choose a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ satisfying $\psi_{n_k}(y_k) > 3\varepsilon'$ for each $k$. Then for each $k$ there exists $m$ with $|\phi_{n_k}|(y_k \wedge m e_{n_k}) > 2\varepsilon'$. Furthermore, by the relation

$$|\phi_{n_k}|(y_k \wedge m e_{n_k}) = \sup\{|\phi_{n_k}(z)| \mid |z| \leq y_k \wedge m e_{n_k}\} \leq$$
$$\leq 2 \cdot \sup\{|\phi_{n_k}(z)| \mid \theta \leq z \leq y_k \vee m e_{n_k}\}$$

choose $z_k \in I_{n_k}$ such that $\theta \leq z_k \leq y_k \wedge m e_{n_k}$ and $|\phi_{n_k}(z_k)| > \varepsilon'$. Then $\{z_k\}$ is a disjoint sequence of $L$, and $\theta \leq \sum_{k=n}^{m} z_k \leq y_n$ for all $m > n$. Hence $z = \sup\{z_n\}$ exists in $L$.

*Part* 3. Now, we define a mapping $\pi : l^\infty \to L$ by $\pi(a) = \sum_{n=1}^{\infty} a_n z_n$ for $a = \{a_n\} \in l^\infty$. The mapping $\pi$ is well-defined and a positive linear mapping such that $\pi(1_n) = z_n$, where $1_n$ is the element of $l^\infty$ with the $n$th coordinate 1 and the remaining coordinates 0. Then the mapping $\pi^{\sim} : L^{\sim} \to (l^\infty)^{\sim}$ is continuous with respect to the topology $\sigma(L^{\sim}, L)$ and $\sigma((l^\infty)^{\sim}, l^\infty)$. Since $\{\phi_{n_k}\}$ is $\sigma(L^{\sim}, L)$-Cauchy sequence, $\{\pi^{\sim}(\phi_{n_k})\}$ is also $\sigma((l^\infty)^{\sim}, l^\infty)$-Cauchy and hence converges to an element $\xi$ of $(l^\infty)^{\sim}$. By Theorem 14.18 of [3] note that $(l^\infty)^{\sim} = l^1 \oplus c_0^{\perp}$. Let $P$ be the projection of $(l^\infty)^{\sim}$ onto $l^1$. Using Theorem 13.14 and Theorem 13.1 of [3] we have $\|P\pi^{\sim}(\phi_{n_k}) - P\xi\|_{l^1} \to 0$ as $k \to \infty$. For $\varepsilon' > 0$ choose $n_0$ such that for all $k > n_0$, $|P\xi(1_k)| < \varepsilon'/2$ and $|P\pi^{\sim}(\phi_{n_k})(1_k) - P\xi(1_k)| < \varepsilon'/2$. Since $P\pi^{\sim}(\phi_{n_k})(1_k) = \pi^{\sim}(\phi_{n_k})(1_k) = \phi_{n_k}(z_k)$ for each $k$, it then follows that $|\phi_{n_k}(z_k)| < \varepsilon'/2 + |P\xi(1_k)| < \varepsilon'$ for all $k \geq n_0$, contradicting the choice of the subsequence $\{\phi_{n_k}\}$ in Part 2. This contradiction completes the proof.

Let $L$ be a Riesz space and $J$ an ideal of $L^{\sim}$. The mapping $x \to \tilde{x}$ of $L$ into $J_n^{\sim}$ defined by $\tilde{x}(\phi) = \phi(x)$ for $\phi \in J$ is a Riesz homomorphism. Denote by $I_J$ the ideal generated by the image of $L$ in $J_n^{\sim}$. As an application of Theorem 7 we have the following results.

COROLLARY 8. Let $L$ be an order-Cauchy complete Riesz space with the s.$\sigma$-order continuity property and let $J$ be an ideal of $L^{\sim}$. Then we have

(1) If $\phi_n \to \theta$ in $J$ for $\sigma(J, L)$, then $\phi_n \to \theta$ $\sigma(J, I_J)$.

(2) Let $P$ be the projection of $J$ onto a band $B$ of $J$. If $\phi_n \to \theta$ in $J$ for $\sigma(J, L)$, then $P\phi_n \to \theta$ $\sigma(J, L)$.

PROOF. (1) If $\{\phi_n\} \subset J$ is $\sigma(J, L)$-convergent to zero, then by Theorem 7 $\{\phi_n\}$ is order-equicontinuous on $L$. Hence the proof is completed by an argument similar to that of Theorem 20.21 given in [1].

(2) Note first that $J$ is a Dedekind complete Riesz space in its own right. Let $\{\phi_n\} \subset J$ and assume that $\phi_n \to \theta$ for $\sigma(J, L)$. Since $J = B \oplus B^{\mathrm{d}}$ and $(J, |\sigma|(J, I_J))' = I_J$, it follows by Theorem 19.5 of [1] that $I_J = B^{\circ} \oplus (B^{\mathrm{d}})^{\circ}$. Hence using (1) the proof follows the argument given in Theorem 20.22 of [1].

The above Theorem 7 extends a result (Theorem 20.20) of Burkinshaw given in [1]. Also, (1) and (2) of the above corollary contain the results (Theorem 20.21 and Corollary 20.22) of Schaefer given in [1], respectively. Furthermore, we have the following.

THEOREM 9. Let $L$ be an order-Cauchy complete Riesz space with the s.$\sigma$-order continuity property. Then every band of $L^{\sim}$ is $\sigma(L^{\sim}, L)$-sequentially complete.

PROOF. Let $B$ be a band of $L^{\sim}$ and $\{\phi_n\}$ a $\sigma(L^{\sim}, L)$-Cauchy sequence of $B$. By Theorem 7 $\{\phi_n\}$ is order-equicontinuous on $L$. Let $I$ be the ideal generated by the image of $L$ into $(L^{\sim})_n^{\sim}$. Then it follows by Theorem 20.16 of [1] that $\{\phi_n\}$ is $\sigma(L^{\sim}, I)$-relatively compact. Let $\phi_0$ be a $\sigma(L^{\sim}, I)$-accumulation point of $\{\phi_n\}$. It is easy to see that $\phi_n \to \phi_0$ for $\sigma(L^{\sim}, L)$. Now, note that $L^{\sim} = B \oplus B^{\mathrm{d}}$ and $I = B^{\circ} \oplus (B^{\mathrm{d}})^{\circ}$. Let $P$ be the projection of $L^{\sim}$ onto $B^{\mathrm{d}}$. Since $\phi_n \to \phi_0$ for $\sigma(L^{\sim}, L)$, it then follows by Corollary 8 that $P\phi_n \to P\phi_0$ for $\sigma(L^{\sim}, L)$, but $P\phi_n = \theta$ for all $n$. Hence $\phi_0 \in B$ holds. Thus $B$ is $\sigma(L^{\sim}, L)$-sequentially complete, and the proof is complete.

We now show that order-equicontinuity is closely related to sequential compactness.

THEOREM 10. Let $L$ be an order-Cauchy complete Riesz space with the s.$\sigma$-order continuity property. Then for a subset $A$ of $L^{\sim}$ the following statements are equivalent:

(1) $A$ is order-equicontinuous on $L$.

(2) For each $\theta \le e \in L$ and sequence $\{\phi_n\}$ of $A$ there exist $\phi \in L^{\sim}$ and a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ satisfying $\lim \phi_{n_k}(x) = \phi(x)$ for all $x \in [\theta, e]$.

PROOF. (1) $\to$ (2) This follows from the same arguments used in the proof of Theorem 20.24 of [1].

(2) $\to$ (1) Note first that $A$ is $\sigma(L^{\sim}, L)$-bounded. Since $L$ is uniformly complete, by Theorem 19.15 of [1] $A$ is also $|\sigma|(L^{\sim}, L)$-bounded. Now, assume that $A$ is not order-equicontinuous on $L$. Then there exist $\varepsilon > 0$, an order bounded disjoint sequence $\{e_n\} \subset [\theta, e]$ and a sequence $\{\phi_n\}$ of $A$ such that $|\phi_n(e_n)| > \varepsilon$. Hence by Part 1 and 2 in the proof of Theorem 7 there exist $\varepsilon' > 0$, a disjoint sequence $\{z_k\}$ of $L^{+}$ and a subsequence $\{\phi_{n_k}\} \subset \{\phi_n\}$ such

that $|\phi_{n_k}(z_k)|>\varepsilon'$ for all $k$ and $z=\sup\{z_k\}$ exists in $L$. From this we have a Riesz isomorphism(into) $\pi: l^\infty \to L$ defined by $\pi(a)=\sum_{k=1}^\infty a_k z_k$ for $a=\{a_n\}\in l^\infty$. By hypothesis let $\{\phi_{m_j}\}$ be a subsequence of $\{\phi_{n_k}\}$ converging to an element $\phi$ in $L^\sim$ on the interval $[\theta, z]$ of $L$. Clearly, $\pi^\sim(\phi_{m_j})$ converges to $\pi^\sim(\phi)$ in $\sigma((l^\infty)^\sim, l^\infty)$. But it then follows by Part 3 in the proof of Theorem 7 that for large $j$,

$$|\pi^\sim(\phi_{m_j})(1_j)-P\pi^\sim(\phi)(1_j)|<\varepsilon'/2 \text{ and } |P\pi^\sim(\phi)(1_j)|<\varepsilon'/2,$$

where $P$ is the projection of $(l^\infty)^\sim$ onto $l^1$. This means that $|\phi_{m_j}(z_j)|<\varepsilon'$ for large $j$, and a contradiction. Thus $A$ is order-equicontinuous on $L$ and the proof is complete.

Let $L$ be a Riesz space and $X$ a Banach space. A linear mapping $T: L\to X$ is called *o-bounded* if $\sup\{\|Tx\| \mid |x|\le e\}<\infty$ holds for any $\theta\le e\in L$. A linear mapping $T: L\to X$ is *o-weakly compact* if $\{Tx \mid |x|\le e\}\subset X$ is $\sigma(X, X^*)$-relatively compact for any $\theta\le e\in L$, where $X^*$ denotes the norm dual of $X$. Let $T: L\to X$ be $o$-bounded and $f\in X^*$. Then the linear form $T^\sim f$ defined by $(T^\sim f)(x)=f(Tx)$ for all $x\in L$ is order bounded and the map $T^\sim: X^*\to L^\sim$ is linear.

In [5] Dodds gave a number of necessary and sufficient conditions for an $o$-bounded map $T: L\to X$ to be $o$-weakly compact. Finally, using Theorem 4.2 of [5] we will generalize the main result (Theorem 2.5) given by Dashiell in [4].

THEOREM 11. Let $L$ be an order-Cauchy complete Riesz space with the s.$\sigma$-order continuity property and $X$ a Banach space. If $T: L\to X$ is $o$-bounded but not $o$-weakly compact, then there exists a subspace $X_0$ of $X$ such that $l^\infty$ is Riesz isomorphic to $T^{-1}(X_0)\subset L$ and isometric to $X_0$.

PROOF. Assume that $T: L\to X$ is $o$-bounded but not $o$-weakly compact. Then by Theorem 4.2 of [5] $\{T^\sim f \mid f\in X_1^*\}$ is not $\sigma(L^\sim, I)$-relatively compact in $L^\sim$, where $X_1^*$ denotes the unit ball of $X^*$ and $I$ is the ideal generated by the image of $L$ into $(L^\sim)_n^\sim$. By Theorem 20.16 and 20.6 of [1] there exist $\varepsilon>0$, an order bounded disjoint sequence $\{e_n\}\subset[\theta, e]$ and a sequence $\{f_n\}$ of $X_1^*$ such that $|(T^\sim f_n)(e_n)|>2\varepsilon$ for each $n$. Hence from the arguments given in Part 1 and 2 of Theorem 7 there exist $\varepsilon'>0$, a disjoint sequence $\{z_n\}$ of $L^+$ and a subsequence $\{f_{n_k}\}\subset\{f_n\}$ such that $|T^\sim f_{n_k}(z_k)|>\varepsilon'/2$ for all $k$ and $z=\sup\{z_k\}$ exists in $L$. Furthermore, by Part 3 of Theorem 7 there exists a Riesz isomorphism $\pi$ from $l^\infty$ into $L$ such that $\pi(a)=\sum_{n=1}^\infty a_n z_n$ for $a=\{a_n\}\in l^\infty$. It then follows that for any $f\in X_1^*$ and $a=\{a_n\}\in l^\infty$,

$$\begin{aligned}|f(T\pi(a))| &\le |T^\sim f|(|\pi(a)|)\\ &=|T^\sim f|(\textstyle\sum |a_n|z_n)\le \|a\|_\infty \sup\{|T^\sim f|(z) \mid f\in X_1^*\}\end{aligned}$$

Hence $T\pi: l^\infty\to X$ is a continuous linear mapping. Also, since

$$\|T\pi(1_k)\|=\|Tz_k\|\ge|(T^\sim f_{n_k})(z_k)|>\varepsilon'/2 \text{ for all } k,$$

the map $T\pi$ satisfies the condition of Rosenthal's theorem (Remark 1 of Proposition 1.2 in [10]). From this there exists some infinite subset $\mathcal{N}'$ of $\mathcal{N}$ such that $T\pi : l^{\infty}(\mathcal{N}') \to X$ is an isomorphism(into). Finally, put $X_0 = T\pi(l^{\infty}(\mathcal{N}'))$. Then it follows immediately that $X_0$ is isometric to $l^{\infty}(\mathcal{N}')$ and $T^{-1}(X_0)$ is Riesz isomorphic to $l^{\infty}(\mathcal{N}')$. The proof of the theorem is now complete.

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